Thiele Formalism: Dynamics of spin textures

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The magnetization texture in a thin nanomagnet can be written as

$$\vec{m}(\vec{r},t) = \vec{m}(\vec{r} - \vec{R}(t)) \tag{1}$$

where $\vec{r}$ is in the plane normal to the thin film normal. Starting with the ansatz that the magnetization texture moves as a whole without changes in the texture allows us to write

$$\dot{\vec{m}} = -(\dot{\vec{R}} \cdot \vec{\nabla})\vec{m} \tag{2}$$

The magnetization dynamics is governed by the LLG equation

$$\dot{\vec{m}} = -\gamma\, \vec{m} \times \vec{H}_{\text{eff}} + \alpha\, \vec{m} \times \dot{\vec{m}} \tag{3}$$

which is equivalent to

$$\vec{m} \times \dot{\vec{m}} = -\gamma\, (\vec{m} \cdot \vec{H}_{\text{eff}})\vec{m} + \gamma \vec{H}_{\text{eff}} - \alpha\, \dot{\vec{m}} \tag{4}$$

Since

$$\vec{m} \cdot \dot{\vec{m}} = 0 \Rightarrow -\dot{R}_j \vec{m} \cdot \partial_j \vec{m} = 0 \tag{5}$$

and since $\dot{R}_j \neq 0$, $\vec{m} \cdot \partial_j \vec{m} = 0$.

Upon substitution of the Thiele ansatz into the LLG

$$-\, \vec{m} \times (\dot{\vec{R}} \cdot \vec{\nabla})\vec{m} = -\gamma\, (\vec{m} \cdot \vec{H}_{\text{eff}})\vec{m} + \gamma \vec{H}_{\text{eff}} + \alpha\, (\dot{\vec{R}} \cdot \vec{\nabla})\vec{m} \tag{6}$$

which is equivalent to

$$-\, \dot{R}_i\, \vec{m} \times \partial_i \vec{m} = -\gamma\, (\vec{m} \cdot \vec{H}_{\text{eff}})\vec{m} + \gamma \vec{H}_{\text{eff}} + \alpha\, \dot{R}_i\, \partial_i \vec{m} \tag{7}$$

where repeated indices are summed over. Taking dot product with $\partial_x \vec{m}$ and $\partial_y \vec{m}$, we get the two equations for the magnetization texture location in the thin magnet,

$$\begin{aligned} -\dot{R}_y\, (\vec{m} \times \partial_y \vec{m}) \cdot \partial_x \vec{m} &= \gamma \vec{H}_{\text{eff}} \cdot \partial_x \vec{m} + \alpha\, \dot{R}_i\, \partial_i \vec{m} \cdot \partial_x \vec{m} \\ -\dot{R}_x\, (\vec{m} \times \partial_x \vec{m}) \cdot \partial_y \vec{m} &= \gamma \vec{H}_{\text{eff}} \cdot \partial_y \vec{m} + \alpha\, \dot{R}_i\, \partial_i \vec{m} \cdot \partial_y \vec{m} \end{aligned} \tag{8}$$

Since $(\vec{m} \times \partial_y \vec{m}) \cdot \partial_x \vec{m} = -(\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m}$ and $(\vec{m} \times \partial_x \vec{m}) \cdot \partial_y \vec{m} = (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m}$,

$$\begin{aligned} \dot{R}_y\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= \gamma \vec{H}_{\text{eff}} \cdot \partial_x \vec{m} + \alpha\, \dot{R}_i\, \partial_i \vec{m} \cdot \partial_x \vec{m} \\ -\dot{R}_x\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= \gamma \vec{H}_{\text{eff}} \cdot \partial_y \vec{m} + \alpha\, \dot{R}_i\, \partial_i \vec{m} \cdot \partial_y \vec{m} \end{aligned} \tag{9}$$

Integrating over the spatial dimension ($L \int d\vec{r}$) and analyzing terms one at a time.

$$\begin{aligned} \dot{R}_y L \int d\vec{r}\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= \gamma L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \partial_x \vec{m} + \alpha\, \dot{R}_i L \int d\vec{r}\, \partial_i \vec{m} \cdot \partial_x \vec{m} \\ -\dot{R}_x L \int d\vec{r}\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= \gamma L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \partial_y \vec{m} + \alpha\, \dot{R}_i L \int d\vec{r}\, \partial_i \vec{m} \cdot \partial_y \vec{m} \end{aligned} \tag{10}$$

For the first term on the right hand side, let us consider the force experienced by the magnetic texture

$$
\begin{aligned}
F_j &= -\frac{dE}{dR_j} = \frac{d}{dR_j} M_s L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \vec{m}(\vec{r}-\vec{R}) \\
&= M_s L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \frac{d\vec{m}}{dR_j} \\
&= -M_s L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \frac{d\vec{m}}{dr_j} \equiv -M_s L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \partial_j \vec{m}
\end{aligned} \tag{11}
$$

since the energy density is $-M_s \vec{H}_{\text{eff}} \cdot \vec{m}(\vec{r}-\vec{R})$. Therefore,

$$
\gamma L \int d\vec{r}\, \vec{H}_{\text{eff}} \cdot \partial_j \vec{m} = -\frac{\gamma}{M_s} F_j \tag{12}
$$

Consequently,

$$
\begin{aligned}
\dot{R}_y \frac{M_s L}{\gamma} \int d\vec{r}\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= -F_x + \alpha \frac{M_s L}{\gamma} \int d\vec{r}\, \partial_x \vec{m} \cdot \partial_i \vec{m}\ \dot{R}_i \\
-\dot{R}_x \frac{M_s L}{\gamma} \int d\vec{r}\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} &= -F_y + \alpha \frac{M_s L}{\gamma} \int d\vec{r}\, \partial_y \vec{m} \cdot \partial_i \vec{m}\ \dot{R}_i
\end{aligned} \tag{13}
$$

Using the definition for topological charge

$$
Q = \frac{1}{4\pi} \int d\vec{r}\, (\partial_x \vec{m} \times \partial_y \vec{m}) \cdot \vec{m} \tag{14}
$$

we can define the Gyrotropic vector

$$
\vec{G} = \hat{z}\, \frac{4\pi M_s L}{\gamma} Q = G_z\, \hat{z} \tag{15}
$$

and defining the dissipation tensor elements as

$$
D_{pq} = \alpha \frac{M_s L}{\gamma} \int d\vec{r}\, \partial_p \vec{m} \cdot \partial_q \vec{m} \tag{16}
$$

With these definitions, the equation of motion for the Skyrmion center is

$$
\begin{aligned}
\dot{R}_y G_z &= -F_x + D_{xi}\, \dot{R}_i \\
-\dot{R}_x G_z &= -F_y + D_{yi}\, \dot{R}_i
\end{aligned} \tag{17}
$$

Therefore in vectorial form, the equation of motion is

$$
\dot{\vec{R}} \times \vec{G} = -\vec{F} + \bar{\bar{D}}\, \dot{\vec{R}} \tag{18}
$$

where $\bar{\bar{D}}$ is the dissipation tensor with elements given by D_{pq}.

A. Solution:

We can easily solve the equation of motion under the action of a linear restoring force in absence of dissipation,

$$
\begin{aligned}
\dot{R}_y G_z &= k R_x \\
-\dot{R}_x G_z &= k R_y
\end{aligned} \tag{19}
$$

Defining a complex variable $Z = R_x + i R_y$,

$$
-i \dot{Z} G_z = k Z \Rightarrow \dot{Z} = i \omega_G Z \tag{20}
$$

Laplace transforming,

$$Z(s) = \frac{Z(0)}{s - i\omega_G} \tag{21}$$

Therefore,

$$Z(t) = Z(0)\exp\left(i\omega_G t\right) \tag{22}$$

which implies

$$\begin{aligned} R_x(t) &= R_x(0)\cos\omega_G t - R_y(0)\sin\omega_G t \\ R_y(t) &= R_x(0)\sin\omega_G t + R_y(0)\cos\omega_G t \end{aligned} \tag{23}$$