Superconductivity Gap Equation

Avinash Rustagi[1, *]

[1]*Department of Physics, North Carolina State University, Raleigh, NC 27695*

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The objective is to show that the gap equation decouples into different channels.

The generalized superconductivity gap equation for a given pairing potential $V(\boldsymbol{k},\boldsymbol{p})$,

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{p}} V(\boldsymbol{k},\boldsymbol{p})\langle c_{-\boldsymbol{p}\downarrow} c_{\boldsymbol{p}\uparrow}\rangle \tag{1}$$

Superconductivity arises from Fermi surface instability to Cooper pair formation, thus it is fair to restrict the electrons in the Gap equation to Fermi surface (i.e. $k = k_F$ and $p = k_F$). We can express the Gap equation in terms of the anomalous Green function $F(\boldsymbol{p}) = \langle c_{-\boldsymbol{p}\downarrow} c_{\boldsymbol{p}\uparrow}\rangle$

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{p}} V(\boldsymbol{k},\boldsymbol{p})F(\boldsymbol{p}) \tag{2}$$

Setting $k = k_F$ and $p = k_F$, the gap equation has only angles ($\hat{k}$ and $\hat{p}$)

$$\Delta(\hat{k}) = -\sum_{\hat{p}} V(\hat{k},\hat{p})F(\hat{p}) \tag{3}$$

Assuming azimuthal symmetry, we can expand the functions $\Delta(\hat{k})$ and $F(\hat{p})$ in spherical harmonics

$$\begin{aligned} \Delta(\hat{k}) &= \sum_{lm} \Delta_l Y_{lm}(\hat{k}) \\ F(\hat{p}) &= \sum_{l'm'} F_{l'} Y_{l'm'}(\hat{p}) \end{aligned} \tag{4}$$

We can use the addition theorem of spherical harmonics to expand the pairing potential

$$V(\boldsymbol{k},\boldsymbol{p}) = \sum_{lm} V_l Y_{lm}(\hat{k}) Y_{lm}(\hat{p})^* \tag{5}$$

Therefore the gap equation in the l-channel can be estimated

$$\begin{aligned} \Delta_l &= \int \frac{d\Omega_k}{4\pi} \Delta(\hat{k}) Y_{lm}(\hat{k})^* \\ &= -\int \frac{d\Omega_p}{4\pi} \int \frac{d\Omega_k}{4\pi} V(\hat{k},\hat{p}) F(\hat{p}) Y_{lm}(\hat{k})^* \\ &= -\int \frac{d\Omega_p}{4\pi} \int \frac{d\Omega_k}{4\pi} \sum_{l'm'} V_{l'} Y_{l'm'}(\hat{k}) Y_{l'm'}(\hat{p})^* \sum_{l''m''} F_{l''} Y_{l''m''}(\hat{p}) Y_{lm}(\hat{k})^* \end{aligned} \tag{6}$$

Using the orthogonality of spherical harmonics

$$\int \frac{d\Omega_p}{4\pi} Y_{lm}(\hat{p})^* Y_{l'm'}(\hat{p}) = \delta_{l,l'} \delta_{m,m'} \tag{7}$$

$$\Delta_l = -\sum_{l'm'} V_{l'} \sum_{l''m''} F_{l''} \delta_{l,l'} \delta_{m,m'} \delta_{l',l''} \delta_{m',m''} \tag{8}$$

which indicates that the different channels decouple

$$\Delta_l = -V_l\, F_l \tag{9}$$

However, we note that we had set all the electron momenta to be at the Fermi surface. In reality, they can have any magnitude within the Debye energy of the Fermi surface,

$$\Delta_l = -\int_0^{\hbar\omega_D} d\xi \, N(0) V_l \, F_l \tag{10}$$

where

$$F_l = \frac{\Delta_l}{\sqrt{\Delta_l^2 + \xi^2}} \qquad \xi = \varepsilon - E_F \tag{11}$$

$$\Delta_l \approx 2\hbar\omega_D e^{-1/N(0)V_l} \tag{12}$$