Skyrmion: Topological Charge

Avinash Rustagi*

The topological charge of a skyrmion is defined as

$$Q_{\rm sk} = \frac{1}{4\pi}\int dxdy\, \vec{m}\cdot\left(\frac{\partial\vec{m}}{\partial x}\times\frac{\partial\vec{m}}{\partial y}\right) \tag{1}$$

The unit magnetization can be described in terms of two fields Θ, Φ:

$$\vec{m} = \begin{pmatrix} \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix} \tag{2}$$

Evaluating the terms in skymrion charge density

$$\begin{aligned} m_x\left(\frac{\partial\vec{m}}{\partial x}\times\frac{\partial\vec{m}}{\partial y}\right)_x &= \sin^3\Theta\cos^2\Phi\left[\frac{\partial\Theta}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial x}\frac{\partial\Theta}{\partial y}\right] \\ m_y\left(\frac{\partial\vec{m}}{\partial x}\times\frac{\partial\vec{m}}{\partial y}\right)_y &= \sin^3\Theta\sin^2\Phi\left[\frac{\partial\Theta}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial x}\frac{\partial\Theta}{\partial y}\right] \\ m_z\left(\frac{\partial\vec{m}}{\partial x}\times\frac{\partial\vec{m}}{\partial y}\right)_z &= \sin\Theta\cos^2\Theta\left[\frac{\partial\Theta}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial x}\frac{\partial\Theta}{\partial y}\right] \end{aligned} \tag{3}$$

Therefore,

$$Q_{\rm sk} = \frac{1}{4\pi}\int dxdy\, \sin\Theta\left[\frac{\partial\Theta}{\partial x}\frac{\partial\Phi}{\partial y}-\frac{\partial\Phi}{\partial x}\frac{\partial\Theta}{\partial y}\right] \tag{4}$$

The transformation from cartesian to polar coordinates

$$\begin{aligned} \frac{\partial\Theta}{\partial x} &= \cos\phi\frac{\partial\Theta}{\partial r}-\frac{\sin\phi}{r}\frac{\partial\Theta}{\partial\phi} \\ \frac{\partial\Theta}{\partial y} &= \sin\phi\frac{\partial\Theta}{\partial r}+\frac{\cos\phi}{r}\frac{\partial\Theta}{\partial\phi} \\ \frac{\partial\Phi}{\partial x} &= \cos\phi\frac{\partial\Phi}{\partial r}-\frac{\sin\phi}{r}\frac{\partial\Phi}{\partial\phi} \\ \frac{\partial\Phi}{\partial y} &= \sin\phi\frac{\partial\Phi}{\partial r}+\frac{\cos\phi}{r}\frac{\partial\Phi}{\partial\phi} \end{aligned} \tag{5}$$

Invoking the observation $\Theta = \Theta(r)$ and $\Phi = \Phi(\phi)$,

$$\begin{aligned} Q_{\rm sk} &= \frac{1}{4\pi}\int_0^\infty dr\,\sin\Theta\frac{d\Theta}{dr}\int_0^{2\pi} d\phi\,\frac{d\Phi}{d\phi} \\ &= \frac{[m_z(0)-m_z(\infty)]}{2}\frac{[\Phi(2\pi)-\Phi(0)]}{2\pi} \end{aligned} \tag{6}$$

For a skyrmion with magnetization in +z direction at the center and -z direction outside the the domain wall implies $m_z(0) - m_z(\infty) = 2$, and with $\Phi(\phi) = \nu\phi + \gamma$ where ν is the vorticity and γ is the helicity,

$$Q_{\rm sk} = \nu \tag{7}$$

which denotes the number of times the magnetization winds i.e. winding number of target space on the base space. For the simplest cases, $\nu = 1$ implying $Q_{\rm sk} = 1$.