Ramsey Interferometry

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Ramsey measurement is done for measuring DC magnetic field. In this measurement, first a $\pi/2$ pulse is applied to an initialized qubit. Then, the qubit is left to evolve freely for a fixed time. Following which, another $\pi/2$ pulse is applied and the state of the qubit is projectively readout.

To begin, let's consider the action of a microwave magnetic field on qubit dynamics. This is important to design the required $\pi/2$ pulse. To do so, let us start with the Hamiltonian for a qubit with an oscillatory magnetic field applied along the x-axis

$$H = \frac{\omega_q}{2}\sigma_{11} - \frac{\omega_q}{2}\sigma_{00} + \Omega\cos(\omega t + \phi)[\sigma_{10} + \sigma_{01}] \tag{1}$$

where $\sigma_{ij} = |i\rangle\langle j|$. We can solve this problem in the interaction picture by expressing the Hamiltonian as

$$\begin{aligned} H &= H_0 + V \\ H_0 &= \frac{\omega}{2}\sigma_{11} - \frac{\omega}{2}\sigma_{00} \\ V &= \frac{\Delta}{2}\sigma_{11} - \frac{\Delta}{2}\sigma_{00} + \Omega\cos(\omega t + \phi)[\sigma_{10} + \sigma_{01}] \end{aligned} \tag{2}$$

Here $\Delta = \omega_q - \omega$ is the detuning between the microwave field and the qubit splitting. We can now move to the interaction picture with the RWA

$$\begin{aligned} H_I &= e^{iH_0 t} V e^{-iH_0 t} \\ &= \frac{\Delta}{2}\sigma_{11} - \frac{\Delta}{2}\sigma_{00} + \frac{\Omega}{2}e^{i\phi}\sigma_{01} + \frac{\Omega}{2}e^{-i\phi}\sigma_{10} \\ &= \frac{1}{2}\begin{bmatrix} -\Delta & \Omega e^{i\phi} \\ \Omega e^{-i\phi} & \Delta \end{bmatrix} \end{aligned} \tag{3}$$

In the interaction picture, the evolution of wavefunction is governed by the interaction Hamiltonian. The eigenvalues and eigenvectors for this Hamiltonian is

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}\tilde{\Omega} \qquad |v_1\rangle = \begin{bmatrix} \cos(\theta/2) \\ -e^{-i\phi}\sin(\theta/2) \end{bmatrix} \\ \lambda_2 &= +\frac{1}{2}\tilde{\Omega} \qquad |v_2\rangle = \begin{bmatrix} \sin(\theta/2) \\ e^{-i\phi}\cos(\theta/2) \end{bmatrix} \end{aligned} \tag{4}$$

where $\tilde{\Omega} = \sqrt{\Omega^2 + \Delta^2}$ and $\tan\theta = \Omega/\Delta$. For now, we will restrict to the case where the driving field phase $\phi = 0$. The wavefunction in the interaction picture evolves as

$$|\Psi_I(t)\rangle = e^{-iH_I t}|\Psi_I(0)\rangle \tag{5}$$

Lets initialize the qubit to state $|\Psi_I(0)\rangle = |0\rangle$,

$$|\Psi_I(0)\rangle = |0\rangle = \cos(\theta/2)|v_1\rangle + \sin(\theta/2)|v_2\rangle \tag{6}$$

Thus,

$$\begin{aligned} |\Psi_I(t)\rangle &= e^{-iH_I t}|\Psi_I(0)\rangle \\ &= \cos(\theta/2)e^{-i\lambda_1 t}|v_1\rangle + \sin(\theta/2)e^{-i\lambda_2 t}|v_2\rangle \end{aligned} \tag{7}$$

Therefore, the probability to find the system in the ground state

$$P_0(t) = |\langle 0|\Psi_I(t)|^2 = \frac{1}{2}[1 + \cos\tilde{\Omega}t] + \frac{\cos^2\theta}{2}[1 - \cos\tilde{\Omega}t] \tag{8}$$

and the transition probability to get to the excited state is

$$P_1(t) = 1 - P_0(t) = \sin^2\theta \sin^2\left(\frac{\tilde{\Omega} t}{2}\right) = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2\left[\frac{\sqrt{\Omega^2+\Delta^2}\,t}{2}\right] \tag{9}$$

The $\pi/2$ pulse is of duration T such that $P_1 = 1/2$,

$$T = \frac{2}{\sqrt{\Omega^2+\Delta^2}}\sin^{-1}\left[\frac{\sqrt{\Omega^2+\Delta^2}}{\Omega\sqrt{2}}\right] \tag{10}$$

The Ramsey pulse can be modeled in terms of the evolution operator as

$$U_R = R_0(T)F(\tau)R_0(T) \tag{11}$$

where

$$R_0(T) = e^{-iH_I T} = e^{i(\Delta T/2)\sigma_z - i(\Omega T/2)\sigma_x} = \begin{bmatrix} \cos\frac{A}{2} + i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} & -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} \\ -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} & \cos\frac{A}{2} - i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} \end{bmatrix} \tag{12}$$

where $A = \tilde{\Omega}T$. And the free evolution operator is

$$F(\tau) = \begin{bmatrix} e^{i\Delta\tau/2} & 0 \\ 0 & e^{-i\Delta\tau/2} \end{bmatrix} \tag{13}$$

Therefore, the total evolution operator for the Ramsey pulse is

$$U_R = \begin{bmatrix} \cos\frac{A}{2} + i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} & -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} \\ -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} & \cos\frac{A}{2} - i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} \end{bmatrix} \begin{bmatrix} \left(\cos\frac{A}{2} + i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2}\right)e^{i\Delta\tau/2} & -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2}e^{i\Delta\tau/2} \\ -i\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2}e^{-i\Delta\tau/2} & \left(\cos\frac{A}{2} - i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2}\right)e^{-i\Delta\tau/2} \end{bmatrix} \tag{14}$$

The probability to find the system in the excited state would be

$$P_1 = |\langle 1|U_R|0\rangle|^2 = |(U_R)_{12}|^2 \tag{15}$$

Thus,

$$(U_R)_{12} = -i\frac{\Omega}{\tilde{\Omega}}\left[\sin A\cos(\Delta\tau/2) - 2\frac{\Delta}{\tilde{\Omega}}\sin^2\frac{A}{2}\sin(\Delta\tau/2)\right] \tag{16}$$

$$\sin^2\frac{A}{2} = \frac{\tilde{\Omega}^2}{2\Omega^2} \qquad \sin A = \frac{\tilde{\Omega}}{\Omega}\sqrt{1-\frac{\Delta^2}{\Omega^2}} \tag{17}$$

Thus,

$$(U_R)_{12} = -i\left[\sqrt{1-\frac{\Delta^2}{\Omega^2}}\cos(\Delta\tau/2) - \frac{\Delta}{\Omega}\sin(\Delta\tau/2)\right] = -i\left[\cos\chi\cos(\Delta\tau/2) - \sin\chi\sin(\Delta\tau/2)\right] = -i\cos(\chi+\Delta\tau/2) \tag{18}$$

where $\sin\chi = \Delta/\Omega$. Therefore

$$P_1 = \cos^2\left(\frac{\Delta\tau}{2} + \chi\right) \tag{19}$$

For the case of general phase in the second $\pi/2$ pulse,

$$U_R = R_\phi(T)F(\tau)R_0(T) \qquad R_\phi(T) = \begin{bmatrix} \cos\frac{A}{2} + i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} & -ie^{i\phi}\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} \\ -ie^{-i\phi}\frac{\Omega}{\tilde{\Omega}}\sin\frac{A}{2} & \cos\frac{A}{2} - i\frac{\Delta}{\tilde{\Omega}}\sin\frac{A}{2} \end{bmatrix} \tag{20}$$

Consequently,

$$P_1 = \cos^2\left(\frac{\Delta\tau}{2} + \chi + \frac{\phi}{2}\right) \tag{21}$$

NUMERICAL RESULTS

Solved using the QuTiP toolbox, the Ramsey fringes can be simulated both in the Schrodinger and interaction picture. The driving field is modeled as

$$\vec{H}_d(t) = \hat{x}\, H_{d,0} \cos\omega t\ [\Theta(T-t) + \Theta(t-T-\tau) - \Theta(t-2T-\tau)] \tag{22}$$

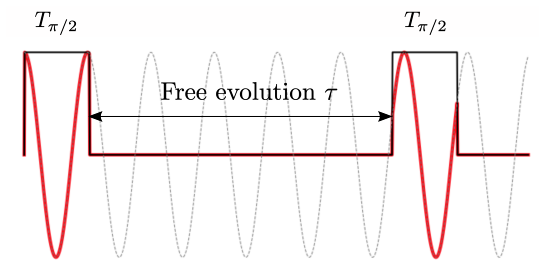

FIG. 1: Pulse protocol

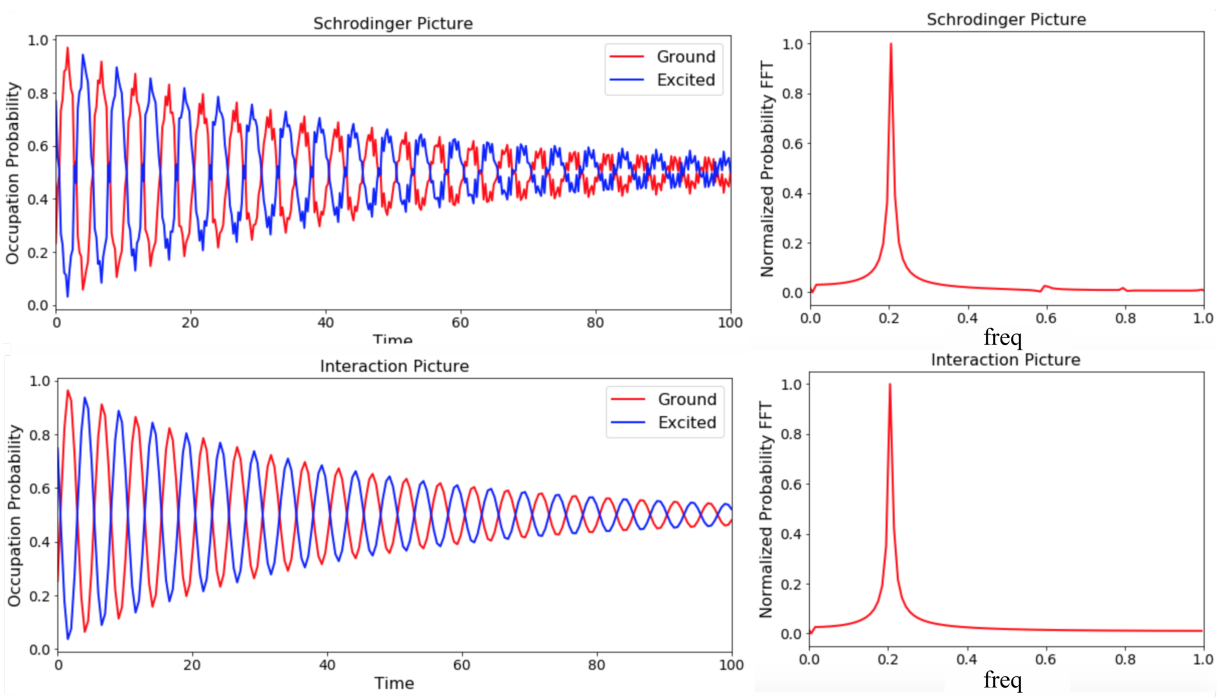

FIG. 2: Ramsey fringes. Parameters $\omega_q = 2\pi \times 1$, $\Delta = 2\pi \times 0.2$, $\omega_d = \omega_q - \Delta$, $\Omega = 2\pi \times 0.4$. As seen, the Ramsey fringes oscillate at a frequency set by the detuning between the qubit splitting and the drive frequency.