Rabi Dynamics

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Rabi dynamics i.e. transitions between the states of a qubit is what allows us to design pulses for accessing states over the Bloch sphere. This is achieved by the application of a microwave magnetic field on the qubit. This is important to design the required $\pi/2$ and π - pulses. To evaluate this, let us start with the Hamiltonian for a qubit with an oscillatory magnetic field applied along the x-axis

$$H = \frac{\omega_q}{2}\sigma_{11} - \frac{\omega_q}{2}\sigma_{00} + \Omega\cos(\omega t + \phi)[\sigma_{10} + \sigma_{01}] \tag{1}$$

where $\sigma_{ij} = |i\rangle\langle j|$. We can solve this problem in the interaction picture by expressing the Hamiltonian as

$$\begin{aligned} H &= H_0 + V \\ H_0 &= \frac{\omega}{2}\sigma_{11} - \frac{\omega}{2}\sigma_{00} \\ V &= \frac{\Delta}{2}\sigma_{11} - \frac{\Delta}{2}\sigma_{00} + \Omega\cos(\omega t + \phi)[\sigma_{10} + \sigma_{01}] \end{aligned} \tag{2}$$

Here $\Delta = \omega_q - \omega$ is the detuning between the microwave field and the qubit splitting. We can now move to the interaction picture with the RWA

$$\begin{aligned} H_I &= e^{iH_0 t} V e^{-iH_0 t} \\ &= \frac{\Delta}{2}\sigma_{11} - \frac{\Delta}{2}\sigma_{00} + \frac{\Omega}{2}e^{i\phi}\sigma_{01} + \frac{\Omega}{2}e^{-i\phi}\sigma_{10} \\ &= \frac{1}{2}\begin{bmatrix} -\Delta & \Omega e^{i\phi} \\ \Omega e^{-i\phi} & \Delta \end{bmatrix} \end{aligned} \tag{3}$$

In the interaction picture, the evolution of wavefunction is governed by the interaction Hamiltonian. The eigenvalues and eigenvectors for this Hamiltonian is

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}\tilde{\Omega} \qquad |v_1\rangle = \begin{bmatrix} \cos(\theta/2) \\ -e^{-i\phi}\sin(\theta/2) \end{bmatrix} \\ \lambda_2 &= +\frac{1}{2}\tilde{\Omega} \qquad |v_2\rangle = \begin{bmatrix} \sin(\theta/2) \\ e^{-i\phi}\cos(\theta/2) \end{bmatrix} \end{aligned} \tag{4}$$

where $\tilde{\Omega} = \sqrt{\Omega^2 + \Delta^2}$ and $\tan\theta = \Omega/\Delta$. For now, we will restrict to the case where the driving field phase $\phi = 0$. The wavefunction in the interaction picture evolves as

$$|\Psi_I(t)\rangle = e^{-iH_I t}|\Psi_I(0)\rangle \tag{5}$$

Lets initialize the qubit to state $|\Psi_I(0)\rangle = |0\rangle$,

$$|\Psi_I(0)\rangle = |0\rangle = \cos(\theta/2)|v_1\rangle + \sin(\theta/2)|v_2\rangle \tag{6}$$

Thus,

$$\begin{aligned} |\Psi_I(t)\rangle &= e^{-iH_I t}|\Psi_I(0)\rangle \\ &= \cos(\theta/2)e^{-i\lambda_1 t}|v_1\rangle + \sin(\theta/2)e^{-i\lambda_2 t}|v_2\rangle \end{aligned} \tag{7}$$

Therefore, the probability to find the system in the ground state

$$P_0(t) = |\langle 0|\Psi_I(t)|^2 = \frac{1}{2}[1 + \cos\tilde{\Omega}t] + \frac{\cos^2\theta}{2}[1 - \cos\tilde{\Omega}t] \tag{8}$$

and the transition probability to get to the excited state is

$$P_1(t) = 1 - P_0(t) = \sin^2\theta\sin^2\left(\frac{\tilde{\Omega}t}{2}\right) = \frac{\Omega^2}{\Omega^2 + \Delta^2}\sin^2\left[\frac{\sqrt{\Omega^2 + \Delta^2}t}{2}\right] \tag{9}$$

Thus, we have evaluated the transition probability as a function of detuning and time.

$$P_1(t) = \frac{1}{1+(\Delta/\Omega)^2}\sin^2\left[\frac{\sqrt{1+(\Delta/\Omega)^2}}{2}\Omega t\right] \tag{10}$$

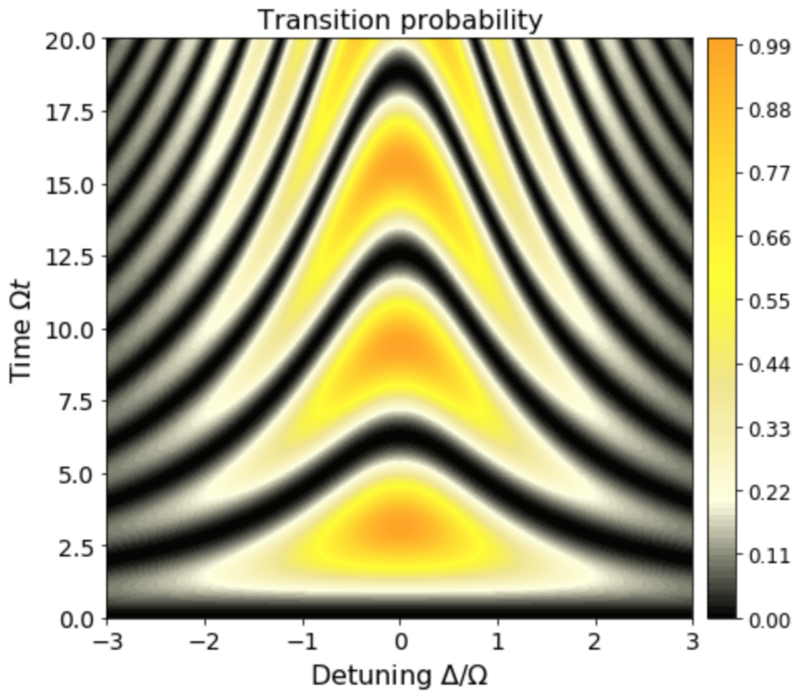

FIG. 1: Transition probability as a function of detuning and pulse duration.

The $\pi/2$ pulse is of duration $T_{\pi/2}$ such that $P_1 = 1/2$,

$$T_{\pi/2} = \frac{2}{\sqrt{\Omega^2+\Delta^2}}\sin^{-1}\left[\frac{\sqrt{\Omega^2+\Delta^2}}{\Omega\sqrt{2}}\right] \tag{11}$$

The π pulse is of duration T_π such that $P_1 = 1$,

$$T_{\pi} = \frac{4}{\sqrt{\Omega^2+\Delta^2}}\sin^{-1}\left[\frac{\sqrt{\Omega^2+\Delta^2}}{\Omega\sqrt{2}}\right] \tag{12}$$