Microscopic - Derivation : Open Quantum System.

$$H = H_S + H_E + H_{int}$$

ρ_{SE} : Total Density Matrix Sys + Env.

Working in interaction picture :

$$\rho_{SE}(t+\Delta t, t) = \rho_{SE}(t) - \frac{i}{\hbar} \int_{t}^{t+\Delta t} dt' \left[H_{int}(t'), \rho_{SE}(t') \right]$$

$$\rho_{SE}(t', t) = \rho_{SE}(t) - \frac{i}{\hbar} \int_{t}^{t'} dt'' \left[H_{int}(t''), \rho_{SE}(t'') \right]$$

Thus upto second order, [Born Approx.]

$$\rho_{SE}(t+\Delta t, t) \approx \rho_{SE}(t) - \frac{i}{\hbar} \int_{t}^{t+\Delta t} dt' \left[H_{int}(t'), \rho_{SE}(t) \right] - \frac{1}{\hbar^2} \int_{t}^{t+\Delta t} dt' \int_{t}^{t'} dt'' \left[H_{int}(t'), \left[H_{int}(t''), \rho_{SE}(t) \right] \right]$$

To get the System dynamics only, we trace out the environment

$$\rho_S(t) = Tr_E \left[\rho_{SE}(t) \right]$$

$$\rho_S(t+\Delta t, t) = \rho_S(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt' \; Tr_E\left[H_{int}(t'), \rho_{SE}(t) \right]$$

$$- \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \; Tr_E\left[H_{int}(t'), \left[H_{int}(t''), \rho_{SE}(t) \right]\right]$$

$$\rho_{SE}(t) = Tr_E[\rho_{SE}] \otimes Tr_S[\rho_{SE}] + \rho_{correlation}$$

Env. has huge # of DOF. Thus, correlations get lost over a time scale τ_c.

[MARKOV] If $\Delta t \gg \tau_c$: ρ_{corr} can be dropped

$$\Rightarrow \quad \rho_{SE}(t) \approx Tr_E[\rho_{SE}] \otimes Tr_S[\rho_{SE}]$$

$$\approx \rho_S(t) \otimes \underbrace{\rho_E(t)}_{\rho_E(0)}$$

$\leftarrow$ Env. is large & does not change

Typical interaction can be written as

$$H_{int} = \sum_{\alpha} \underset{\substack{\uparrow \\ \text{System} \\ \text{Operators}}}{\hat{S}_{\alpha}} \otimes \underset{\substack{\uparrow \\ \text{Reservoir operators}}}{\hat{R}_{\alpha}}$$

and we can assume that $\langle R_{\alpha} \rangle = \mathrm{Tr}_E [\rho_E R_{\alpha}] = 0$

No mean field in Env.

$$\Rightarrow \quad \mathrm{Tr}_E \left[H_{int}(t'), \rho_S(t) \otimes \rho_E(0) \right] = 0$$

Therefore,

$$\rho_S(t+\Delta t, t) \approx \rho_S(t) - \frac{1}{\hbar^2} \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \; \mathrm{Tr}_E \left[H_{int}(t'), \left[H_{int}(t''), \rho_S \otimes \rho_E \right] \right]$$

To proceed further, we consider an example :

TLS in a bath of photons.

TLS in photon bath.

$$H = H_A + H_B + H_{int}$$

$$H_A = \frac{\hbar\omega_0}{2}\sigma_z$$

$$H_B = \sum_{k\mu} \hbar\omega_k \, a^\dagger_{k\mu} a_{k\mu}$$

$$H_{int} = \hbar \sum_{k\mu} \left[g_{k\mu} a_{k\mu} \sigma_+ + g^*_{k\mu} \sigma_- a^\dagger_{k\mu} \right]$$

$\uparrow$

in rotating wave approx.

In interaction picture,

$$a_{k\mu}(t) = e^{iH_0 t} a_{k\mu} e^{-iH_0 t} = a_{k\mu} e^{-i\omega_k t}$$

$$\sigma_+(t) = e^{iH_0 t} \sigma_+ e^{-iH_0 t} = \sigma_+ e^{i\omega_0 t}$$

$$H_{int}(t') = \hbar \sum_{k\mu} \left[g_{k\mu} a_{k\mu} \sigma_+ e^{i(\omega_0 - \omega_k)t'} + g^*_{k\mu} a^\dagger_{k\mu} \sigma_- e^{-i(\omega_0-\omega_k)t'} \right]$$

$$= F(t')\sigma_+ e^{i\omega_0 t'} + F^\dagger(t')\sigma_- e^{-i\omega_0 t'}$$

$$F(t') = \hbar \sum_{k\mu} g_{k\mu} a_{k\mu} e^{-i\omega_k t'}$$

$$F^\dagger(t') = \hbar \sum_{k\mu} g^*_{k\mu} a^\dagger_{k\mu} e^{i\omega_k t'}$$

Thus,

$$\rho_S(t+\Delta t, t) \approx \rho(t) - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \; Tr_E \Big[H_{int}(t')\, H_{int}(t'')\, \rho_S \otimes \rho_E$$

$$- H_{int}(t')\, \rho_S \otimes \rho_E\, H_{int}(t'')$$

$$- H_{int}(t'')\, \rho_S \otimes \rho_E\, H_{int}(t')$$

$$+ \rho_S \otimes \rho_E\, H_{int}(t'')\, H_{int}(t') \Big]$$

$$\langle a^\dagger_{k\mu}\, a_{k'\mu'} \rangle = Tr_E \left[\rho_E\, a^\dagger_{k\mu}\, a_{k'\mu'} \right] = \bar{n}_{k\mu}\, \delta_{kk'}\, \delta_{\mu\mu'}$$

$$\langle a_{k'\mu'}\, a^\dagger_{k\mu} \rangle = (\bar{n}_{k\mu} + 1)\, \delta_{kk'}\, \delta_{\mu\mu'}$$

1) $$Tr_E\left[H_{int}(t')\, H_{int}(t'')\, \rho_S \otimes \rho_E \right] = \sigma_+ \sigma_- \rho_S\, e^{i\omega_0(t'-t'')} \langle F(t')\, F^\dagger(t'') \rangle$$
$$+ \sigma_- \sigma_+ \rho_S\, e^{-i\omega_0(t'-t'')} \langle F^\dagger(t')\, F(t'') \rangle$$

2) $$Tr_E\left[H_{int}(t')\, \rho_S \otimes \rho_E\, H_{int}(t'') \right] = \sigma_+ \rho_S \sigma_-\, e^{i\omega_0(t'-t'')} \langle F^\dagger(t'')\, F(t') \rangle$$
$$+ \sigma_- \rho_S \sigma_+\, e^{-i\omega_0(t'-t'')} \langle F(t'')\, F^\dagger(t') \rangle$$

3) $$Tr_E\left[H_{int}(t'')\, \rho_S \otimes \rho_E\, H_{int}(t') \right] = \sigma_+ \rho_S \sigma_-\, e^{-i\omega_0(t'-t'')} \langle F^\dagger(t')\, F(t'') \rangle$$
$$+ \sigma_- \rho_S \sigma_+\, e^{i\omega_0(t'-t'')} \langle F(t')\, F^\dagger(t'') \rangle$$

$$4)\quad Tr_E\left[\rho_S \otimes \rho_E \; H_{int}(t'')\, H_{int}(t')\right] = \rho_S\,\sigma_+\sigma_-\; e^{-i\omega_0(t'-t'')} \; \langle F(t'')\, F^\dagger(t')\rangle$$

$$+\; \rho_S\,\sigma_-\sigma_+\; e^{i\omega_0(t'-t'')}\; \langle F^\dagger(t'')\, F(t')\rangle$$

$$K_1 = \langle F(t_1)\, F^\dagger(t_2)\rangle = \hbar^2 \sum_{\substack{k_1\mu_1\\ k_2\mu_2}} g_{k_1\mu_1}\, g^*_{k_2\mu_2} \;\langle a_{k_1\mu_1}\, a^\dagger_{k_2\mu_2}\rangle\; e^{-i\omega_{k_1}t_1}\; e^{i\omega_{k_2}t_2}$$

$$= \hbar^2 \sum_{k\mu} |g_{k\mu}|^2 \left[\bar{n}_{k\mu}+1\right] e^{-i\omega_k(t_1-t_2)}$$

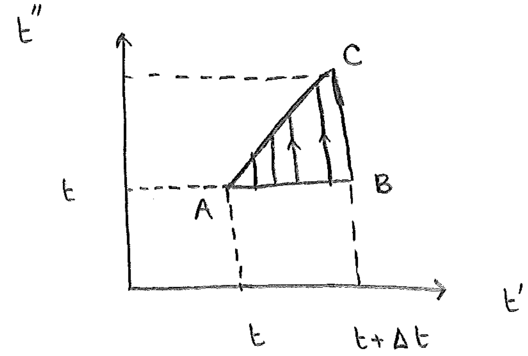
$$K_2 = \langle F^\dagger(t_1)\, F(t_2)\rangle = \hbar^2 \sum_{k\mu} |g_{k\mu}|^2\; \bar{n}_{k\mu}\; e^{i\omega_k(t_1-t_2)}$$

$$1) \quad = \sigma_+\sigma_-\rho_S\; e^{i\omega_0(t'-t'')}\; K_1(t'-t'') + \sigma_-\sigma_+\rho_S\; e^{-i\omega_0(t'-t'')}\; K_2(t'-t'')$$

$$2) \quad = \sigma_+\rho_S\sigma_-\; e^{i\omega_0(t'-t'')}\; K_2(t''-t') + \sigma_-\rho_S\sigma_+\; e^{-i\omega_0(t'-t'')}\; K_1(t''-t')$$

$$3) \quad = \sigma_+\rho_S\sigma_-\; e^{-i\omega_0(t'-t'')}\; K_2(t'-t'') + \sigma_-\rho_S\sigma_+\; e^{i\omega_0(t'-t'')}\; K_1(t'-t'')$$

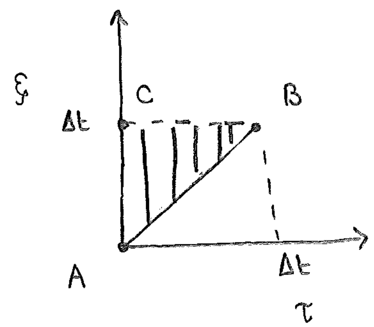
$$4) \quad = \rho_S\sigma_+\sigma_-\; e^{-i\omega_0(t'-t'')}\; K_1(t''-t') + \rho_S\sigma_-\sigma_+\; e^{i\omega_0(t'-t'')}\; K_2(t''-t')$$



$$\int_t^{t+\Delta t} dt' \int_t^{t'} dt''$$

$$\tau = t' - t'' \qquad \tau(A) = 0 \qquad \tau(B) = \Delta t \qquad \tau(C) = 0$$

$$\xi = t' - t \qquad \xi(A) = 0 \qquad \xi(B) = \Delta t \qquad \xi(C) = \Delta t$$



$$\int_0^{\Delta t} d\tau \int_\tau^{\Delta t} d\xi$$

$$\Rightarrow \quad \Delta\rho_s = -\frac{1}{\hbar^2} \int_0^{\Delta t} d\tau \int_\tau^{\Delta t} d\xi \left\{ (\sigma_+\sigma_-\rho_s - \sigma_-\rho_s\sigma_+)\, e^{i\omega_0\tau}\, K_1(\tau) + (\sigma_-\sigma_+\rho_s - \sigma_+\rho_s\sigma_-)\, e^{-i\omega_0\tau}\, K_2(\tau) \right\}$$

$$+ \ h.c$$

$K(\tau)$ falls off rapidly after τ_c

$\Delta t \gg \tau_c$ [Markov] $\leftarrow$ δ - correlated

$$\int_\tau^{\Delta t} d\xi \approx \int_{\tau_c}^{\Delta t} d\xi \approx \int_0^{\Delta t} d\xi$$

$$\Rightarrow \quad \frac{\Delta\rho_S}{\Delta t} \approx -\frac{1}{\hbar^2}\int_0^{\Delta t} d\tau \Big\{ (\sigma_+\sigma_-\rho_S - \sigma_-\rho_S\sigma_+)\, e^{i\omega_0\tau} K_1(\tau)$$

$$+ (\sigma_-\sigma_+\rho_S - \sigma_+\rho_S\sigma_-)\, e^{-i\omega_0\tau} K_2(\tau)$$

$$+ (\rho_S\sigma_+\sigma_- - \sigma_-\rho_S\sigma_+)\, e^{-i\omega_0\tau} K_1(-\tau)$$

$$+ (\rho_S\sigma_-\sigma_+ - \sigma_+\rho_S\sigma_-)\, e^{i\omega_0\tau} K_2(-\tau) \Big\}$$

$$\int_0^{\Delta t\to\infty} d\tau \; e^{i\omega_0\tau} \underset{\substack{\downarrow \\ \text{decays over } \tau_c}}{K_1(\tau)} = \hbar^2 \sum_{k\mu} |g_{k\mu}|^2 (\bar{n}_{k\mu}+1) \underbrace{\int_0^\infty d\tau \; e^{-i(\omega_k-\omega_0)\tau}}_{\pi\,\delta(\omega_0-\omega_k) - i\,P\left(\frac{1}{\omega_k-\omega_0}\right)}$$

since $\Delta t \gg \tau_c$

$$= \hbar^2 \int_0^\infty d\omega_k \; \mathcal{D}(\omega_k)\, \overline{|g(\omega_k)|}^2\, (\bar{n}_{k\mu}+1)$$

$$\times \left\{ \pi\,\delta(\omega_0-\omega_k) - i\,P\left(\frac{1}{\omega_k-\omega_0}\right) \right\}$$

$$\equiv \frac{\Gamma}{2}(\bar{n}+1) - \frac{i}{\hbar}\,\delta E_{\bar{n}+1}$$

$$\Downarrow$$

light-shift $\quad P\displaystyle\int d\omega_k \frac{(\bar{n}+1)\,|g(\omega_k)|^2}{\omega_k-\omega_0}$

In Markov Approx :

$$\frac{d\rho_S}{dt} = -\frac{i}{\hbar}\left[\delta H_{\text{light-shift}}, \rho_S\right]$$

$$+ \frac{\Gamma}{2}(\bar{n}+1)\left[2\sigma_-\rho_S\sigma_+ - \rho_S\sigma_+\sigma_- - \sigma_+\sigma_-\rho_S\right]$$

$$+ \frac{\Gamma}{2}\bar{n}\left[2\sigma_+\rho_S\sigma_- - \rho_S\sigma_-\sigma_+ - \sigma_-\sigma_+\rho_S\right]$$

Compare to Lindblad form :

$$L_{abs} = \sqrt{\Gamma\bar{n}}\;\sigma_+$$

$$L_{emiss} = \sqrt{\Gamma(\bar{n}+1)}\;\sigma_-$$