Landau-Lifshitz-Gilbert - Fokker-Planck Equation

Avinash Rustagi*
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Landau-Lifshitz-Gilbert equation (equivalent to the Landau-Lifshitz equation with renormalized parameters) governs the magnetization dynamics in an effective magnetic field $\vec{B}$ determined by the free energy landscape. It is simple to generalize it to include a fluctuating field $\vec{b}(t)$.

$$\begin{aligned}
\frac{d\vec{M}}{dt} &= -\gamma_G \vec{M} \times (\vec{B} + \vec{b}(t)) + \frac{\alpha}{M_s}\vec{M} \times \frac{d\vec{M}}{dt} \\
&= -\gamma_G \vec{M} \times (\vec{B} + \vec{b}(t)) + \frac{\alpha}{M_s}\vec{M} \times \left[-\gamma_G \vec{M} \times (\vec{B} + \vec{b}(t)) + \frac{\alpha}{M_s}\vec{M} \times \frac{d\vec{M}}{dt} \right] \\
&= -\gamma_G \vec{M} \times (\vec{B} + \vec{b}(t)) - \gamma_G \frac{\alpha}{M_s}\vec{M} \times \vec{M} \times (\vec{B} + \vec{b}(t)) + \frac{\alpha^2}{M_s^2}\vec{M} \times \vec{M} \times \frac{d\vec{M}}{dt} \\
&= -\gamma_G \vec{M} \times (\vec{B} + \vec{b}(t)) - \gamma_G \frac{\alpha}{M_s}\vec{M} \times \vec{M} \times (\vec{B} + \vec{b}(t)) - \alpha^2 \frac{d\vec{M}}{dt} \\
&= -\frac{\gamma_G}{1+\alpha^2}\vec{M} \times (\vec{B} + \vec{b}(t)) - \frac{\alpha\gamma_G}{M_s(1+\alpha^2)}\vec{M} \times \vec{M} \times (\vec{B} + \vec{b}(t))
\end{aligned} \tag{1}$$

Defining $\gamma = \gamma_G/(1+\alpha^2)$:

$$\frac{d\vec{M}}{dt} = -\gamma \vec{M} \times (\vec{B} + \vec{b}(t)) - \frac{\alpha\gamma}{M_s}\vec{M} \times \vec{M} \times (\vec{B} + \vec{b}(t)) \tag{2}$$

where the fluctuating field $\vec{b}(t)$ is a Gaussian stochastic variable

$$\langle b_i(t) \rangle = 0 \qquad \langle b_i(t_1) b_j(t_2) \rangle = 2D\delta_{ij}\delta(t_1 - t_2) \tag{3}$$

Assuming that damping is weak, we can drop the fluctuating field from the second term in the LLG equation

$$\frac{d\vec{M}}{dt} = -\gamma \vec{M} \times (\vec{B} + \vec{b}(t)) - \frac{\alpha\gamma}{M_s}\vec{M} \times \vec{M} \times \vec{B} \tag{4}$$

Considering the Fokker-Planck equation for the probability density of having magnetization $\vec{M}$ at time t for a given realization of stochastic field is

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(\vec{M},t) &= -\frac{\partial}{\partial \vec{M}} \cdot [\dot{\vec{M}}\rho(\vec{M},t)] \\
&= -\frac{\partial}{\partial \vec{M}} \cdot \left[\left(-\gamma \vec{M} \times (\vec{B} + \vec{b}(t)) - \frac{\alpha\gamma}{M_s}\vec{M} \times \vec{M} \times \vec{B} \right) \rho(\vec{M},t) \right] \\
&= \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\gamma \vec{M} \times \vec{B} + \frac{\alpha\gamma}{M_s}\vec{M} \times \vec{M} \times \vec{B} \right) \rho(\vec{M},t) \right] + \frac{\partial}{\partial \vec{M}} \cdot \left[\gamma \vec{M} \times \vec{b}(t)\rho(\vec{M},t) \right] \\
&= \partial_{M_i} \left[\left(\gamma\epsilon_{ijk}M_j B_k + \frac{\alpha\gamma}{M_s}\epsilon_{ijk}M_j\epsilon_{kpq}M_p B_q \right) \rho(\vec{M},t) \right] + \partial_{M_i} \left[\gamma\epsilon_{ijk}M_j b_k(t)\rho(\vec{M},t) \right] \\
&= \left[\gamma\epsilon_{ijk}M_j B_k \partial_{M_i} + \frac{\alpha\gamma}{M_s}\epsilon_{ijk}\epsilon_{kpq}M_j\partial_{M_i}M_p B_q \right] \rho(\vec{M},t) + [\gamma\epsilon_{ijk}M_j b_k(t)\partial_{M_i}]\,\rho(\vec{M},t) \\
&= -L_0\rho(\vec{M},t) - L_1\rho(\vec{M},t)
\end{aligned} \tag{5}$$

where the operators L_0 and L_1 are

$$\begin{aligned}
L_0 &= -\left[\gamma\epsilon_{ijk}M_j B_k \partial_{M_i} + \frac{\alpha\gamma}{M_s}\epsilon_{ijk}\epsilon_{kpq}M_j\partial_{M_i}M_p B_q \right] \\
L_1 &= -\left[\gamma\epsilon_{ijk}M_j b_k(t)\partial_{M_i} \right]
\end{aligned} \tag{6}$$

To get to the observable probability density, we need to average over the various realizations of the random force $\xi(t)$

$$P(\vec{M},t) = \langle \rho(\vec{M},t) \rangle_b \tag{7}$$

To evaluate this average, define

$$\rho(\vec{M},t) = e^{-L_0 t} \sigma(\vec{M},t) \tag{8}$$

which implies

$$\frac{\partial}{\partial t}\sigma(\vec{M},t) = -e^{L_0 t} L_1 e^{-L_0 t}\sigma(\vec{M},t) \equiv -V(t)\sigma(\vec{M},t) \tag{9}$$

The formal solution to this equation is

$$\sigma(\vec{M},t) = \exp\left[-\int_0^t dt_1\, V(t_1)\right]\sigma(\vec{M},t=0) \tag{10}$$

Averaging over the random force realizations

$$\langle \sigma(\vec{M},t)\rangle_\xi = \langle \exp\left[-\int_0^t dt_1\, V(t_1)\right]\rangle_b \sigma(\vec{M},t=0) \tag{11}$$

which upon using the cumulant expansion relation

$$\langle e^{-i\Phi(t)}\rangle = \exp\left[\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}c_n\right] \tag{12}$$

gives (assuming that the random force is Gaussian implying that only second cumulant is non-zero equivalent to stating that only even moments are non-zero)

$$\langle \sigma(\vec{M},t)\rangle_b = \exp\left[\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \langle V(t_1)V(t_2)\rangle_b\right]\sigma(\vec{M},t=0) \tag{13}$$

Thus we evaluate the average in the exponential

$$\begin{aligned}\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \langle V(t_1)V(t_2)\rangle_b &= \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \langle e^{L_0 t_1}\gamma\epsilon_{ijk}M_j b_k(t_1)\partial_{M_i}e^{-L_0 t_1}e^{L_0 t_2}\gamma\epsilon_{pqr}M_q b_r(t_2)\partial_{M_p}e^{-L_0 t_2}\rangle_b \\ &= \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \gamma^2\langle b_k(t_1)b_r(t_2)\rangle_b e^{L_0 t_1}\epsilon_{ijk}M_j\partial_{M_i}e^{-L_0 t_1}e^{L_0 t_2}\epsilon_{pqr}M_q\partial_{M_p}e^{-L_0 t_2} \\ &= \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2\, \gamma^2 2D\delta_{kr}\delta(t_1-t_2)e^{L_0 t_1}\epsilon_{ijk}M_j\partial_{M_i}e^{-L_0 t_1}e^{L_0 t_2}\epsilon_{pqr}M_q\partial_{M_p}e^{-L_0 t_2} \\ &= D\gamma^2\int_0^t dt_1\, e^{L_0 t_1}\epsilon_{ijk}M_j\partial_{M_i}\epsilon_{pqk}M_q\partial_{M_p}e^{-L_0 t_1}\end{aligned} \tag{14}$$

where we have used the Gaussian nature of the random field i.e. $\langle b_k(t_1)b_r(t_2)\rangle = 2D\delta_{kr}\delta(t_1-t_2)$. Thus

$$\langle \sigma(\vec{M},t)\rangle_b = \exp\left[D\gamma^2\int_0^t dt_1\, e^{L_0 t_1}\epsilon_{ijk}M_j\partial_{M_i}\epsilon_{pqk}M_q\partial_{M_p}e^{-L_0 t_1}\right]\sigma(\vec{M},0) \tag{15}$$

Taking the time-derivative of the above equation

$$\frac{\partial}{\partial t}\langle \sigma(\vec{M},t)\rangle_b = D\gamma^2 e^{L_0 t}\epsilon_{ijk}M_j\partial_{M_i}\epsilon_{pqk}M_q\partial_{M_p}e^{-L_0 t}\langle \sigma(\vec{M},t)\rangle_b \tag{16}$$

which translates to

$$\frac{\partial}{\partial t}\langle \rho(\vec{M},t)\rangle_b = -L_0\langle \rho(\vec{M},t)\rangle_b + D\gamma^2\epsilon_{ijk}M_j\partial_{M_i}\epsilon_{pqk}M_q\partial_{M_p}\langle \rho(\vec{M},t)\rangle_b \tag{17}$$

which is the Fokker-Planck equation in terms of the macroscopic probability density

$$\frac{\partial}{\partial t}P(\vec{M},t) = -L_0 P(\vec{M},t) + D\gamma^2 \epsilon_{ijk} M_j \partial_{M_i} \epsilon_{pqk} M_q \partial_{M_p} P(\vec{M},t) \tag{18}$$

which can be simplified as

$$\epsilon_{ijk} M_j \partial_{M_i} \epsilon_{pqk} M_q \partial_{M_p} = \partial_{M_i} \epsilon_{ijk} M_j \epsilon_{kpq} M_q \partial_{M_p} = -\partial_{M_i} \epsilon_{ijk} M_j \epsilon_{kqp} M_q \partial_{M_p} \equiv -\frac{\partial}{\partial \vec{M}} \cdot \left[\vec{M} \times \vec{M} \times \frac{\partial}{\partial \vec{M}}\right] \tag{19}$$

and

$$-L_0 P(\vec{M},t) = \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\gamma \vec{M} \times \vec{B} + \frac{\alpha\gamma}{M_s} \vec{M} \times \vec{M} \times \vec{B}\right) P(\vec{M},t)\right] \tag{20}$$

Therefore, the final form of the Fokker Planck equation is

$$\begin{aligned} \frac{\partial}{\partial t}P(\vec{M},t) &= \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\gamma \vec{M} \times \vec{B} + \frac{\alpha\gamma}{M_s} \vec{M} \times \vec{M} \times \vec{B}\right) P(\vec{M},t)\right] - D\gamma^2 \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\vec{M} \times \vec{M} \times \frac{\partial}{\partial \vec{M}}\right) P(\vec{M},t)\right] \\ &= \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\gamma \vec{M} \times \vec{B} + \frac{\alpha\gamma}{M_s} \vec{M} \times \vec{M} \times \vec{B}\right) P(\vec{M},t) - D\gamma^2 \left(\vec{M} \times \vec{M} \times \frac{\partial}{\partial \vec{M}}\right) P(\vec{M},t)\right] \end{aligned} \tag{21}$$

Thermal Equilibrium

In absence of an external force and in thermal equilibrium ($\partial_t P = 0$), the probability distribution is given by the Boltzmann factor

$$P_0 \propto e^{-\beta\mathcal{H}} \Rightarrow \frac{\partial}{\partial \vec{M}} P_0 = \beta \vec{B} P_0 \tag{22}$$

where $\mathcal{H}$ is the free energy and thus the effective field is

$$\vec{B} = -\frac{\partial \mathcal{H}}{\partial \vec{M}} \tag{23}$$

This means that the quantity $\gamma(\vec{M} \times \vec{B})P_0(\vec{M})$ is divergence-less i.e.

$$\begin{aligned} \frac{\partial}{\partial \vec{M}} \cdot \left[(\vec{M} \times \vec{B})P_0(\vec{M})\right] &= \left[\frac{\partial}{\partial \vec{M}} \cdot (\vec{M} \times \vec{B})\right] P_0(\vec{M}) + (\vec{M} \times \vec{B}) \cdot \frac{\partial}{\partial \vec{M}} P_0(\vec{M}) \\ &= \left[\frac{\partial}{\partial \vec{M}} \cdot (\vec{M} \times \vec{B})\right] P_0(\vec{M}) + \beta(\vec{M} \times \vec{B}) \cdot \vec{B} P_0 \\ &= 0 \end{aligned} \tag{24}$$

Hence, from the Fokker-Planck equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\frac{\alpha\gamma}{M_s} \vec{M} \times \vec{M} \times \vec{B}\right) P_0(\vec{M}) - D\gamma^2 \left(\vec{M} \times \vec{M} \times \frac{\partial}{\partial \vec{M}}\right) P_0(\vec{M})\right] \\ &= \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\frac{\alpha\gamma}{M_s} \vec{M} \times \vec{M} \times \vec{B}\right) P_0(\vec{M}) - \beta D\gamma^2 \left(\vec{M} \times \vec{M} \times \vec{B}\right) P_0(\vec{M})\right] \end{aligned} \tag{25}$$

which implies

$$D = \frac{\alpha k_B T}{\gamma M_s} \tag{26}$$

Thus

$$\langle b_i(t_1) b_j(t_2) \rangle = \frac{2\alpha k_B T}{\gamma M_s} \delta_{ij} \delta(t_1 - t_2) \tag{27}$$

Diffusion Timescale

To consider the timescale corresponding to pure diffusion, we can set the external effective field to zero $\vec{B} = 0$. Therefore, the Fokker Planck equation takes the simplified form

$$\frac{\partial}{\partial t} P(\vec{M}, t) = -D\gamma^2 \frac{\partial}{\partial \vec{M}} \cdot \left[\left(\vec{M} \times \vec{M} \times \frac{\partial}{\partial \vec{M}} \right) P(\vec{M}, t) \right] \tag{28}$$

It is clear that the only timescale in this pure diffusion process happens to be

$$t_D^{-1} = D\gamma^2 = \frac{\alpha k_B T}{\gamma M_s} \gamma^2 = \frac{\alpha \gamma k_B T}{M_s} = \frac{\alpha \gamma_G k_B T}{M_s(1+\alpha^2)} \tag{29}$$

MATHEMATICAL RELATIONS

Moments and Characteristic Function

Probability distribution functions (PDF) are normalized:

$$\int_{-\infty}^{\infty} dx\, P(x) = 1 \tag{30}$$

which implies that the Fourier component of PDF at $k = 0$ is unity. The Fourier transform of the PDF can be defined as

$$P(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, P(x) \tag{31}$$

and from the normalization condition $P(k = 0) = 1$. The function $P(k)$ is referred to as the “Characteristic Function”. The moments of the PDF can be thereby expressed in terms of the derivatives of the Characteristic Function.

$$\begin{aligned}
m_1 &= \langle x \rangle = \int_{-\infty}^{\infty} dx\, x\, P(x) = i \frac{\partial P(k)}{\partial k}\bigg|_{k=0} \\
m_2 &= \langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2\, P(x) = i^2 \frac{\partial^2 P(k)}{\partial k^2}\bigg|_{k=0} \\
&\vdots \\
m_n &= \langle x^n \rangle = \int_{-\infty}^{\infty} dx\, x^n\, P(x) = i^n \frac{\partial^n P(k)}{\partial k^n}\bigg|_{k=0}
\end{aligned} \tag{32}$$

Therefore

$$P(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n \tag{33}$$

Cumulants and Cumulant Generating Function

From the relation between the PDF and the characteristic function

$$P(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, P(k) \tag{34}$$

a “Cumulant Generating Function” $\psi(k)$ is defined as

$$P(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{\psi(k)} \tag{35}$$

where $\psi(k) = \text{Log}[P(k)]$ is the function whose Taylor series coefficients at the origin $k = 0$ are the "Cumulants".

$$c_n = \frac{1}{i^n} \frac{\partial^n \psi(k)}{\partial k^n}\bigg|_{k=0} \tag{36}$$

Therefore

$$\begin{aligned} \psi(k) &= -ikc_1 - \frac{1}{2!}k^2 c_2 .... \\ &= \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n \end{aligned} \tag{37}$$

Comparing to the Characteristic function expansion in terms of moments

$$\psi(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n = \text{Log}\left[\sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n\right] \tag{38}$$

implies

- $c_1 = m_1$ which is the "Mean"
- $c_2 = m_2 - m_1^2 = \sigma^2$ which is the "Variance" [σ: Standard Deviation]
- $c_3 = m_3 - 3m_1m_2 + 2m_1^3$ which is the "Skewness"
- $c_4 = m_4 - 3m_2^2 - 4m_1m_3 + 12m_1^2m_2 - 6m_1^4$ which is the "Kurtosis"

Therefore

$$P(k) = \exp\left[\sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n\right] = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n \tag{39}$$

which implies

$$P(k=1) = \exp\left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} c_n\right] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} m_n \tag{40}$$

Consider the following average

$$\begin{aligned} \langle e^{-i\Phi(t)} \rangle &= \langle \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \Phi(t)^n \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \langle \Phi(t)^n \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} m_n \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} c_n\right] \end{aligned} \tag{41}$$