Kramers Kronig Relations

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I. RETARDED

Causality means that the cause precedes the effects. A simple model case is the retarded Green function

$$G^R_k(t-t') = -i\theta(t-t')\langle\{c_k(t), c^\dagger_k(t')\}\rangle = -i\theta(t-t')e^{-i\varepsilon_k(t-t')} \tag{1}$$

In Fourier space

$$\begin{aligned} G^R(\omega) &= \int_{-\infty}^{\infty} d(t-t')\, G^R_k(t-t')e^{i\omega(t-t')} \\ &= -i\int_0^{\infty} d(t-t')\, e^{i(\omega-\varepsilon_k)(t-t')} \end{aligned} \tag{2}$$

To make the integral converge, we can introduce a small positive infinitesimal factor δ

$$\begin{aligned} G^R(\omega) &= -i\int_0^{\infty} d(t-t')\, e^{i(\omega-\varepsilon_k)(t-t')}e^{-\delta(t-t')} \\ &= -i\int_0^{\infty} d(t-t')\, e^{i(\omega-\varepsilon_k+i\delta)(t-t')} \\ &= \frac{1}{\omega-\varepsilon_k+i\delta} \end{aligned} \tag{3}$$

The retarded Green function has poles in the lower half of the complex plane and is analytic in the upper half.

Consider a function χ_R with a pole on the real axis and is analytic in the upper half of the complex plane. Then by Cauchy's theorem

$$\oint_C dz\, \frac{\chi_R(z)}{z-\omega_0} = 0 \tag{4}$$

where the contour C does not include the singularity.

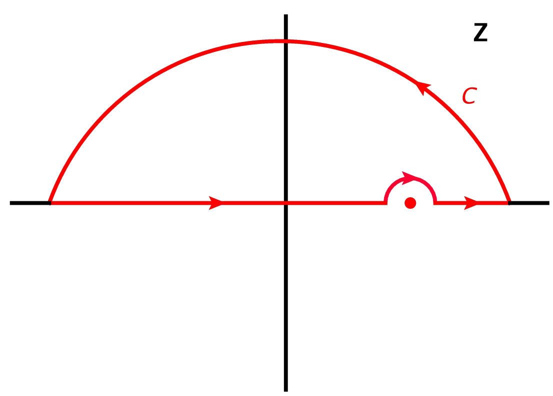

FIG. 1. Contour for Retarded function that is analytic in the entire upper half plane.

$$\oint_C dz\,\frac{\chi_R(z)}{z-\omega_0} = 0$$
$$\lim_{\epsilon\to 0}\int_{-\infty}^{\omega_0-\epsilon} d\omega\,\frac{\chi_R(\omega)}{\omega-\omega_0} + \lim_{\epsilon\to 0}\int_{\omega_0-\epsilon}^{\infty} d\omega\,\frac{\chi_R(\omega)}{\omega-\omega_0} - i\pi\chi_R(\omega_0) = 0 \tag{5}$$
$$\Rightarrow \quad \mathcal{P}\int_{-\infty}^{\infty} d\omega\,\frac{\chi_R(\omega)}{\omega-\omega_0} = i\pi\chi_R(\omega_0)$$

The function $\chi_R = \chi'_R + i\chi''_R$ is complex. Thus comparing the real and imaginary parts

$$\begin{aligned}\chi'_R(\omega_0) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega\,\frac{\chi''_R(\omega)}{\omega-\omega_0}\\ \chi''_R(\omega_0) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega\,\frac{\chi'_R(\omega)}{\omega-\omega_0}\end{aligned} \tag{6}$$

II. ADVANCED

Anti-Causality means that the effect precedes the cause. A simple model case is the advanced Green function

$$G_k^A(t-t') = i\theta(t'-t)\langle\{c_k(t), c_k^\dagger(t')\}\rangle = i\theta(t'-t)e^{-i\varepsilon_k(t-t')} \tag{7}$$

In Fourier space

$$\begin{aligned}G^A(\omega) &= \int_{-\infty}^{\infty} d(t-t')\, G_k^A(t-t')e^{i\omega(t-t')}\\ &= i\int_{-\infty}^{0} d(t-t')\, e^{i(\omega-\varepsilon_k)(t-t')}\end{aligned} \tag{8}$$

To make the integral converge, we can introduce a small positive infinitesimal factor δ

$$\begin{aligned}G^A(\omega) &= i\int_{-\infty}^{0} d(t-t')\, e^{i(\omega-\varepsilon_k)(t-t')}e^{\delta(t-t')}\\ &= i\int_{-\infty}^{0} d(t-t')\, e^{i(\omega-\varepsilon_k-i\delta)(t-t')}\\ &= \frac{1}{\omega-\varepsilon_k-i\delta}\end{aligned} \tag{9}$$

The advanced Green function has poles in the upper half of the complex plane and is analytic in the lower half.
Consider a function χ_A with a pole on the real axis and is analytic in the lower half of the complex plane. Then by

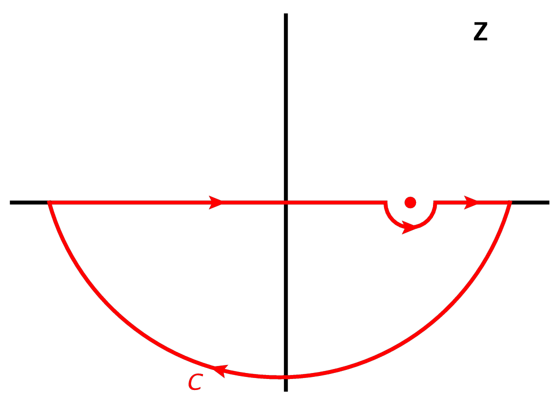

FIG. 2. Contour for Advanced function that is analytic in the entire lower half plane.

Cauchy's theorem

$$\oint_C dz\,\frac{\chi_A(z)}{z-\omega_0} = 0 \tag{10}$$

where the contour C does not include the singularity.

$$
\begin{aligned}
&\oint_C dz\, \frac{\chi_A(z)}{z-\omega_0} = 0 \\
&\lim_{\epsilon\to 0}\int_{-\infty}^{\omega_0-\epsilon} d\omega\, \frac{\chi_A(\omega)}{\omega-\omega_0} + \lim_{\epsilon\to 0}\int_{\omega_0-\epsilon}^{\infty} d\omega\, \frac{\chi_A(\omega)}{\omega-\omega_0} + i\pi\chi_A(\omega_0) = 0 \\
\Rightarrow\quad &\mathcal{P}\int_{-\infty}^{\infty} d\omega\, \frac{\chi_A(\omega)}{\omega-\omega_0} = -i\pi\chi_A(\omega_0)
\end{aligned} \tag{11}
$$

The function $\chi_A = \chi'_A + i\chi''_A$ is complex. Thus comparing the real and imaginary parts

$$
\begin{aligned}
\chi'_A(\omega_0) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega\, \frac{\chi''_A(\omega)}{\omega-\omega_0} \\
\chi''_A(\omega_0) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega\, \frac{\chi'_A(\omega)}{\omega-\omega_0}
\end{aligned} \tag{12}
$$