Fokker-Planck Equation

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Fokker-Planck equation is a widely used equation that describes the time evolution of the probability of a distribution of Brownian particles that is subject to random forces. Such an equation can be derived in two steps:

1) Equation of motion for the probability density $\rho(x,v,t)$ to find the Brownian particle in an interval $(x, x+dx)$ and $(v, v+dv)$ at time t for one realization of the random force $\xi(t)$.

2) Average over many realizations of the random force to obtain the macroscopically observed probability density $P(x,v,t) = \langle \rho(x,v,t) \rangle_\xi$.

Consider the phase space (x,v) and the probability to find the particle in an interval $(x, x+dx)$ and $(v, v+dv)$ at time t is given $\rho(x,v,t)dxdv$. Since the total number of particles is conserved over the entire phase space

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, \rho(x,v,t) = 1 \tag{1}$$

Now if we consider the rate of change of particles in volume V_0 of the phase space that has a surface S, this rate of change is equal to the outflow of particles through the surface S_0. Thus by continuity

$$\frac{\partial}{\partial t} \int_{V_0} dV\, \rho(x,v,t) = -\int_{S_0} d\vec{S}.\dot{\vec{x}}\, \rho(x,v,t) \tag{2}$$

where $\dot{\vec{x}} = (\dot{x}, \dot{v})$ is the velocity in phase space [dot represents the time derivative]. This is simply the "Continuity Equation" in phase space. By Gauss' Theorem

$$\int_{S_0} d\vec{S}.\dot{\vec{x}}\, \rho(x,v,t) = \int_{V_0} dV\, \vec{\nabla} \cdot [\dot{\vec{x}}\, \rho(x,v,t)] \tag{3}$$

where $\vec{\nabla} = (\partial_x, \partial_v)$. Therefore

$$\frac{\partial}{\partial t}\rho(x,v,t) = -\frac{\partial}{\partial x}[\dot{x}\rho(x,v,t)] - \frac{\partial}{\partial v}[\dot{v}\rho(x,v,t)] \tag{4}$$

since we can arbitrarily choose the volume V_0 in the phase space.

For Brownian motion of a particle in a potential $V(x)$ providing force $F(x) = -\partial_x V(x)$,

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\gamma \frac{v}{m} + \frac{F(x)}{m} + \frac{\xi(t)}{m} \end{aligned} \tag{5}$$

therefore

$$\begin{aligned} \frac{\partial}{\partial t}\rho(x,v,t) &= -\frac{\partial}{\partial x}[\dot{x}\rho(x,v,t)] - \frac{\partial}{\partial v}[\dot{v}\rho(x,v,t)] \\ &= -\frac{\partial}{\partial x}\left[v\rho(x,v,t)\right] + \frac{\partial}{\partial v}\left[\gamma \frac{v}{m}\rho(x,v,t)\right] - \frac{\partial}{\partial v}\left[\frac{F(x)}{m}\rho(x,v,t)\right] - \frac{\partial}{\partial v}\left[\frac{\xi(t)}{m}\rho(x,v,t)\right] \\ &= -L_0\rho(x,v,t) - L_1\rho(x,v,t) \end{aligned} \tag{6}$$

where the operators L_0 and L_1 are

$$\begin{aligned} L_0 &= v\frac{\partial}{\partial x} - \frac{\gamma}{m} - \frac{\gamma}{m} v \frac{\partial}{\partial v} + \frac{F(x)}{m}\frac{\partial}{\partial v} \\ L_1 &= \frac{\xi(t)}{m}\frac{\partial}{\partial v} \end{aligned} \tag{7}$$

To get to the observable probability density, we need to average over the various realizations of the random force $\xi(t)$

$$P(x, v, t) = \langle \rho(x, v, t) \rangle_\xi \tag{8}$$

To evaluate this average, define

$$\rho(x, v, t) = e^{-L_0 t} \sigma(x, v, t) \tag{9}$$

which implies

$$\frac{\partial}{\partial t} \sigma(x, v, t) = -e^{L_0 t} L_1 e^{-L_0 t} \sigma(x, v, t) \equiv -V(t) \sigma(x, v, t) \tag{10}$$

The formal solution to this equation is

$$\sigma(x, v, t) = \exp\left[-\int_0^t dt_1\, V(t_1)\right] \sigma(x, v, 0) \tag{11}$$

Averaging over the random force realizations

$$\langle \sigma(x, v, t) \rangle_\xi = \langle \exp\left[-\int_0^t dt_1\, V(t_1)\right] \rangle_\xi \sigma(x, v, 0) \tag{12}$$

which upon using the cumulant expansion relation

$$\langle e^{-i\Phi(t)} \rangle = \exp\left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} c_n\right] \tag{13}$$

gives (assuming that the random force is Gaussian implying that only second cumulant is non-zero equivalent to stating that only even moments are non-zero)

$$\langle \sigma(x, v, t) \rangle_\xi = \exp\left[\frac{1}{2} \int_0^t dt_1 \int_0^t dt_2\, \langle V(t_1) V(t_2) \rangle_\xi\right] \sigma(x, v, 0) \tag{14}$$

Thus we evaluate the average in the exponential

$$\begin{aligned}
\frac{1}{2} \int_0^t dt_1 \int_0^t dt_2\, \langle V(t_1) V(t_2) \rangle_\xi &= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2\, \langle e^{L_0 t_1} \frac{\xi(t_1)}{m} \frac{\partial}{\partial v} e^{-L_0 t_1} e^{L_0 t_2} \frac{\xi(t_2)}{m} \frac{\partial}{\partial v} e^{-L_0 t_2} \rangle_\xi \\
&= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2\, \frac{\langle \xi(t_1) \xi(t_2) \rangle_\xi}{m^2} e^{L_0 t_1} \frac{\partial}{\partial v} e^{-L_0 t_1} e^{L_0 t_2} \frac{\partial}{\partial v} e^{-L_0 t_2} \\
&= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2\, \frac{g \delta(t_1 - t_2)}{m^2} e^{L_0 t_1} \frac{\partial}{\partial v} e^{-L_0 t_1} e^{L_0 t_2} \frac{\partial}{\partial v} e^{-L_0 t_2} \\
&= \frac{1}{2} \int_0^t dt_1\, \frac{g}{m^2} e^{L_0 t_1} \frac{\partial^2}{\partial v^2} e^{-L_0 t_1}
\end{aligned} \tag{15}$$

where we have used the Gaussian nature of the random force i.e. $\langle \xi(t_1) \xi(t_2) \rangle_\xi = g \delta(t_1 - t_2)$. Thus

$$\langle \sigma(x, v, t) \rangle_\xi = \exp\left[\frac{g}{2m^2} \int_0^t dt_1\, e^{L_0 t_1} \frac{\partial^2}{\partial v^2} e^{-L_0 t_1}\right] \sigma(x, v, 0) \tag{16}$$

Taking the time-derivative of the above equation

$$\frac{\partial}{\partial t} \langle \sigma(x, v, t) \rangle_\xi = \frac{g}{2m^2} e^{L_0 t} \frac{\partial^2}{\partial v^2} e^{-L_0 t} \langle \sigma(x, v, t) \rangle_\xi \tag{17}$$

which translates to

$$\frac{\partial}{\partial t} \langle \rho(x, v, t) \rangle_\xi = -L_0 \langle \rho(x, v, t) \rangle_\xi + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \langle \rho(x, v, t) \rangle_\xi \tag{18}$$

which is the Fokker-Planck equation in terms of the macroscopic probability density

$$\frac{\partial}{\partial t}P(x,v,t) = -v\frac{\partial}{\partial x}P(x,v,t) + \frac{\partial}{\partial v}\left[\left(\frac{\gamma}{m}v - \frac{F(x)}{m}\right)P(x,v,t)\right] + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}P(x,v,t) \tag{19}$$

In absence of an external force and in thermal equilibrium ($\partial_t P = 0$), the probability distribution is given by the Boltzmann factor

$$P_0 \propto e^{-\beta m v^2/2} \Rightarrow \partial_v P_0 = -\beta m v P_0 \tag{20}$$

Hence

$$\begin{aligned}
0 &= \frac{\partial}{\partial v}\left[\frac{\gamma}{m}vP_0(v)\right] + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}P_0(v) \\
0 &= \frac{\partial}{\partial v}\left[\frac{\gamma}{m}vP_0(v)\right] - \frac{g}{2m^2}\frac{\partial}{\partial v}\beta m v P_0 \\
0 &= \frac{\partial}{\partial v}\left[\left(\frac{\gamma}{m} - \frac{\beta g}{2m}\right)vP_0(v)\right]
\end{aligned} \tag{21}$$

which implies

$$g = 2\gamma k_B T \tag{22}$$

MATHEMATICAL RELATIONS

Moments and Characteristic Function

Probability distribution functions (PDF) are normalized:

$$\int_{-\infty}^{\infty} dx\, P(x) = 1 \tag{23}$$

which implies that the Fourier component of PDF at $k = 0$ is unity. The Fourier transform of the PDF can be defined as

$$P(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, P(x) \tag{24}$$

and from the normalization condition $P(k = 0) = 1$. The function $P(k)$ is referred to as the "Characteristic Function". The moments of the PDF can be thereby expressed in terms of the derivatives of the Characteristic Function.

$$\begin{aligned}
m_1 &= \langle x \rangle = \int_{-\infty}^{\infty} dx\, x\, P(x) = i\frac{\partial P(k)}{\partial k}\bigg|_{k=0} \\
m_2 &= \langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2\, P(x) = i^2\frac{\partial^2 P(k)}{\partial k^2}\bigg|_{k=0} \\
&\vdots \\
m_n &= \langle x^n \rangle = \int_{-\infty}^{\infty} dx\, x^n\, P(x) = i^n\frac{\partial^n P(k)}{\partial k^n}\bigg|_{k=0}
\end{aligned} \tag{25}$$

Therefore

$$P(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n \tag{26}$$

Cumulants and Cumulant Generating Function

From the relation between the PDF and the characteristic function

$$P(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, P(k) \tag{27}$$

a "Cumulant Generating Function" $\psi(k)$ is defined as

$$P(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{\psi(k)} \tag{28}$$

where $\psi(k) = \text{Log}[P(k)]$ is the function whose Taylor series coefficients at the origin $k = 0$ are the "Cumulants".

$$c_n = \frac{1}{i^n} \frac{\partial^n \psi(k)}{\partial k^n}\bigg|_{k=0} \tag{29}$$

Therefore

$$\begin{aligned} \psi(k) &= -ikc_1 - \frac{1}{2!}k^2 c_2.... \\ &= \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n \end{aligned} \tag{30}$$

Comparing to the Characteristic function expansion in terms of moments

$$\psi(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n = \text{Log}\left[\sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n\right] \tag{31}$$

implies

- $c_1 = m_1$ which is the "Mean"
- $c_2 = m_2 - m_1^2 = \sigma^2$ which is the "Variance" [σ: Standard Deviation]
- $c_3 = m_3 - 3m_1 m_2 + 2m_1^3$ which is the "Skewness"
- $c_4 = m_4 - 3m_2^2 - 4m_1 m_3 + 12m_1^2 m_2 - 6m_1^4$ which is the "Kurtosis"

Therefore

$$P(k) = \exp\left[\sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} c_n\right] = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} m_n \tag{32}$$

which implies

$$P(k=1) = \exp\left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} c_n\right] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} m_n \tag{33}$$

Consider the following average

$$\begin{aligned} \langle e^{-i\Phi(t)} \rangle &= \langle \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \Phi(t)^n \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \langle \Phi(t)^n \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} m_n \\ &= \exp\left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} c_n\right] \end{aligned} \tag{34}$$